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AN INVARIANT FINITE-DIMENSIONAL APPROXIMATION TO THE NAVIER-STOKES EQUATIONS AND SELF-EXCITED OSCILLATORY MODES OF POISEUILLE FLOW*

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A method of determining the finite-dimensional projection of the Navier-Stokes equations onto invariant attractive manifolds is developed. In particular, the projection on a two-dimensional manifold yields an amplitude equation analogous to that discussed by L.D. Landau in /1/.

$$\frac{dA}{dt} = A\left(\gamma + \sum_{n=1}^{\infty} b_{2n}A^{2n}\right) \tag{0.1}$$

with an analytic series in powers of A. The method makes feasible the study of the development of non-linear quasiperiodic perturbations and the occurence of stochastic modes in specific hydrodynamic flows.

It is well-known that the characteristics of the initial stage of turbulence are, in many cases, determined by the behaviour of non-linear perturbations in the initial flow. Beginning with /1/, the basic subject of the study of the non-linear theory of hydrodynamic stability was the equation for the perturbation amplitude (0.1). A method of determining the coefficient b_2 for plane-parallel flows at low values of γ, was proposed in /2/, and developed further in /3, 4/.

In the case of another class of non-linear perturbations /5/ replacement of the time made possible the summation of the infinite series in the amplitude equation and gave an explicit dependence of the perturbation amplitude on time. Analysis of the domain of existence of these solutions /6/ has shown that they can be realized when $|\gamma| > 0$.

There is a close connection between the non-linear theory of hydrodynamic stability and the theory of bifurcation of the Navier-Stokes equation /7-9/. The amplitudes of unstable periodic modes of the theory of bifurcation correspond to the threshold modes for the initial perturbations, and the stable periodic modes describe the limit flows formed by increasing perturbations. The conditions for the existence and uniqueness of periodic modes are obtained in the theory of bifurcations, and a method is developed for determining the amplitudes of selfexcited oscillations in the form of analytic series. Numerical computations based on this method /10, 11/ gave a rigorous proof of the subcritical instability of Poiseuille flow.

An asymptotic method was developed in /12/, making it possible to determine any coefficients b_{2n} in Eq.(0.1). Numerical determination of the coefficients b_3 and b_4 has shown /6, 13/ that in the subcritical region of Poiseuille flow a stable selfexcited oscillatory mode may exist side by side with the unstable mode, the increasing non-linear perturbations reaching a stable mode.

1. *Invariant attractive manifolds.*

We shall consider the flow of a viscous incompressible fluid in a bounded region Ω with the boundary $\partial\Omega$. The velocity of the flow $v(x, t)$ and the pressure $p(x, t)$ are determined by the Navier-Stokes equations

$$\partial v/\partial t + (v\cdot\nabla)v = -\nabla p + \nu\Delta v + f, \quad \nabla\cdot v = 0 \tag{1.1}$$

We shall assume that stationary solutions (v_0, p_0) of system (1.1) exists under the given boundary conditions, and we shall seek solutions of the form

$$v(x, t) = v_0(x) + u(x, t), \quad p(x, t) = p_0(x) + q(x, t)$$

We introduce the following Hilbert spaces:

$$H = \{u \in [L_2(\Omega)]^3;\ \nabla\cdot u = 0,\ u\cdot n\,|_{\partial\Omega} = 0\}$$
$$K = \{u \in [W_2^1(\Omega)]^3;\ \nabla\cdot u = 0,\ u\cdot n\,|_{\partial\Omega} = 0\}$$
$$T = \{u \in [W_2^2(\Omega)]^3;\ \nabla\cdot u = 0,\ u\,|_{\partial\Omega} = 0\}$$

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where $W_l^m(\Omega)$ are Sobolev spaces. Then (see e.g. /14/) the problem of determining (u, q) will be reduced to a differential equation with closed unbounded operators in the Hilbert space H

$$du/dt = -L_\nu u + N(u;\nu) \tag{1.2}$$
$$(L_\nu u = -\Pi(\nu\Delta u - v_0\cdot\nabla u - u\cdot\nabla v_0),\quad N(u) = -\Pi u\cdot\nabla u)$$

where L_ν is a linear operator with the domain of definition T, Π is an orthogonal operator in H in $[L_2(\Omega)]^3$. The non-linear operator N acts on T in K, and in the present case it is independent of the parameter ν.

Let us consider the general case of Eq.(1.2) in some Hilbert space. We shall assume that the operators L_ν, N satisfy the following conditions:

1°. $N(0;\nu) = N_u(0;\nu) = 0,\quad \nu \in V \subset R^m$.

2°. The domain of definition $D(L)$ of the operator L_ν is independent of ν.

3°. The eigenvalues L_ν lie within the sector $|\arg(\zeta + a_\nu)| \leqslant \theta < \pi/2, a_\nu \geqslant 0$, and the following estimate holds for the resolvent $R_\nu(\zeta)$ of the operator $(-L_\nu)$:

$$\|R_\nu(\zeta)\| \equiv \|(L_\nu + \zeta)^{-1}\| \leqslant M_\varepsilon/|\zeta - a_\nu|,\quad |\arg(\zeta - a_\nu)| \leqslant \pi - \theta - \varepsilon,\ \forall\varepsilon > 0$$

From condition 3° it follows that the operator $(-L_\nu)$ is a generating operator of an analytic semigroup /15/. Moreover, there exists a fractional power of the operator $(-L_\nu, -a_\nu)$: $(-L_\nu - a_\nu)^\alpha,\ 0 < \alpha < 1,\ D[(-L_\nu - a_\nu)^\alpha] \supset D(L_\nu)$.

Let us introduce the norms

$$\|u\|_1 = \|u\| + \|(L_\nu + a_\nu)u\|,\quad u \in D(L)$$
$$\|u\|_\alpha = \|u\| + \|(-L_\nu - a_\nu)^\alpha u\|,\quad u \in D[(-L_\nu - a_\nu)^\alpha]$$

Then the sets $D(L)$ and $D[(-L_\nu - a_\nu)^\alpha]$ will become Banach spaces D and D_α respectively.

4°. The operator $N(u;\nu)$ has a domain of definition independent of ν, and represents a mapping of D onto D_α at some α. The mapping has a derivative $N_u(u;\nu)$, which satisfies the Lipshits condition.

5°. The spectrum $\sigma(u)$ of the operator $(-L_\nu)$ admits of the separation $\sigma(\nu) = \sigma_1(\nu) \cup \sigma_2(\nu),\ \sigma_1(\nu) \cap \sigma_2(\nu) = \varnothing$, where $\sigma_1(\nu)$ is the bounded part of the spectrum ($\sigma_1(\nu)$ lies within a closed curve).

6°. The following inequalities exist:

$$\varkappa_\nu \geqslant \delta_1 > 0,\quad \varkappa_\nu - q_\nu \geqslant \delta_2 > 0,\quad \nu \in V$$
$$(\varkappa_\nu = -\sup_{\zeta\in\sigma_2(\nu)} \operatorname{Re}\zeta,\ q_\nu = -\inf_{\zeta\in\sigma_1(\nu)} \operatorname{Re}\zeta)$$

where δ_1, δ_2 are constants.

All conditions 1°-6° hold for the Navier-Stokes equations, and $\alpha < {}^1/_4$.

We shall study the classical solutions of the initial problem for Eq.(1.2)

$$u(0) = u_0 \in D(L) \tag{1.3}$$
$$u \in C^0(0,\infty; D(L)) \cap C^1(0,\infty; H)$$

According to condition 5° there exists a projection operator P_ν, such that the space H can be represented in the form of a direct sum of orthogonal subspaces

$$H = P_\nu H \oplus (I - P_\nu)H$$

The operator $(-L_1) = -P_\nu L_\nu$ exists in the subspace $P_\nu H$, is bounded, and its spectrum is equal to $\sigma_1(\nu)$. The spectrum of the unbounded operator $(-L_2) = -(I - P_\nu)L_\nu$ is equal to $\sigma_2(\nu)$. Eq.(1.2) is equivalent to the following system of equations:

$$dy/dt = -L_1 y + N_1(y + z;\nu),\quad dz/dt = -L_2 z + N_2(y + z;\nu) \tag{1.4}$$
$$u = y + z,\quad y = P_\nu u,\quad z = (I - P_\nu)u;\quad N_1 = P_\nu N,\quad N_2 = (I - P_\nu)N$$

We shall call the graph of the function $z = Z(y)$ $(M = \{y, Z(y)\})$ the locally invariant manifold (LIM) $M \subset D$ of Eq.(1.2) such, that

$$u(t) = y(t) + Z(y(t)), \quad 0 \leqslant t \leqslant t_0$$

provided that $u(0) = y(0) + Z(y(0))$, and $\|y\|_1 < \rho$, $0 \leqslant t \leqslant t_0$ (ρ is a constant).

Theorem 1. Let conditions 1°-6° hold. Then Eq.(1.2) has the following LIM:

$$M = \{(y, z);\ z = Z(y),\ Z(0) = 0\}$$

The function $Z(y)$ is defined in the sphere $\|y\|_1 < \rho$ and satisfies the Lipschitz condition and $\|Z(y)\|_1 < \rho$.

We obtain positive quantities b_1, b_2, such that the following limit holds:

$$\rho + \rho^{1/\alpha} < \begin{cases} b_1(\varkappa_\nu - q_\nu), & q_\nu \geqslant 0 \\ b_2\varkappa_\nu, & q_\nu < 0 \end{cases}$$

Any global solution of problem (1.2), (1.3) in which

$$\|P_\nu u(t)\|_1 < \rho, \quad t \in [0, \infty); \quad \|(I - P_\nu) u(0)\|_1 < \rho$$

will be attracted by the manifold *M*, i.e. $\gamma > 0$, can be found such, that

$$\|Z(P_\nu u(t)) - (I - P_\nu \mid u(t))\|_1 \leqslant \text{const}\,\|Z(y_0) - z_0\|_1 \exp(-\gamma t)$$
$$t \to \infty, \quad y_0 = P_\nu u(0), \quad z_0 = (I - P_\nu)\, u(0)$$

A scheme for constructing the proof is described below* . (*For the complete proof see: Skobelev B.Yu. Analytic projection of non-linear evolutionary equations onto finite-dimensional invariant manifolds. Preprint No.22-86. Novosibirsk, Inst. Theor. Applied Mechanics, Siberian Branch, Academy of Sciences of the USSR, 1986.)

We carry out the change of variable and operator $y \to \rho y$, $z \to \rho z$ $(\rho > 0)$ and $N \to \bar{N}$, in Eqs. (1.4), where

$$\bar{N}(y + z;\ \nu) = N_\rho(\chi(y) + z;\ \nu), \quad N_\rho(u;\ \nu) = \rho^{-1} N(\rho u;\ \nu)$$

$\chi(y)$ is a normalizing function in $P_\nu D$: $\chi(y) = y$, with $\|y\|_1 \leqslant 1$

$$\|\chi(y)\|_1 \leqslant k_1, \quad \|\chi(y_1) - \chi(y_2)\|_1 \leqslant k\|y_1 - y_2\|_1$$

We shall call the integral manifold of transformed system (1.4) a set in the product $[0, \infty) \times D$ consisting of integral curves of this system /16/. Let us discuss the integral manifolds $\bar{M}(z_0)$ describing by equations of the type

$$z = f(t, y;\ z_0), \quad \|f(t, y_1;\ z_0) - f(t, y_2;\ z_0)\|_1 \leqslant \eta\|y_1 - y_2\|_1$$

The manifold $\bar{M}(z_0)$ contains integral curves with fixed initial value $z = z_0$. When ρ and $z = f(t, y; z_0)$ are given, the first equation of the transformed system will have a unique global solution

$$y(t) = Y(t, \tau, y;\ f), \quad t, \tau \in [0, \infty)$$

satisfying the condition $y(\tau) = y$. The second equation of transformed system (1.4) yields the following integrofunctional equation for determining $f(\tau, y; z_0)$:

$$f(\tau, y; z_0) = V_2(\tau) z_0 + \int_0^\tau V_2(\tau - t)\,\bar{N}_2[Y(t, \tau, y; f) + f(t, Y(t, \tau, y; f); z_0]\,dt, \quad V_2(t) = \exp(-L_2 t), \quad 0 \leqslant t < \infty \tag{1.5}$$

The principle of compressed mappings is used to prove the existence and uniqueness of the solution of Eq.(1.5), under the condition that the parameter ρ satisfies the inequality quoted in the theorem and z_0 belongs to the sphere $\|z_0\|_1 < 1$. In the same manner we will prove that $\|f\|_1 < 1$ when $t > t_0(z_0)$, and the manifolds $M(z_0)$ with different z_0 converge to each other exponentially as $t \to \infty$. In the next stage we establish that all manifolds $\bar{M}(z_0)$ tend to a unique limit manifold $\bar{M}$, which is invariant for the transformed system (1.4). From the methods used in constructing the operator $\bar{N}$ we see that for every trajectory of system (1.4) with $\|y\|_1 < \rho$ we have a corresponding trajectory of the transformed system with $\|y\|_1 < 1$. Therefore, the presence of an invariant manifold $\bar{M}$ guarantees the existence of a locally invariant manifold *M* for Eq.(1.2). In the general case the manifold *M* is not unique, because of the arbitrariness in the choice of the normalizing transformation $\chi(y)$.

From Theorem 1 it follows that we can reduce the study of the trajectories of the evolutionary equation with unbounded operators (1.2) to the analysis of the trajectories of an evolutionary equation with bounded operators L_1 and $N_1(y+Z(y))$. If on the other hand the spectrum of σ_1 consists of a finite number of isolated eigenvalues, the problem will be reduced to the study of a finite-dimensional dynamic system.

Other results concerning LIM are given in /17, 18/.

2. ***Analytic projection on finite-dimensional invariant manifolds.*** In order to find the projection of the initial evolutionary equation onto an invariant manifold (IM) in the course of solving specific problems, we shall formulate additional conditions for the operators L_ν and N.

7°. The non-linear operator $N(u;\nu)$ is an analytic operator acting from $D\times V$ into H.

8°. The vector $L_\nu u$ depends analytically on $\nu\in V$ for any $u\in D$.

9°. The bounded part of the spectrum $\sigma_1(\nu)$ consists of n pairs of simple, isolated eigenvalues $(\lambda_1,\bar\lambda_i)$

$$\lambda_i(\nu)=\gamma_i(\nu)+i\omega_i(\nu)\quad (i=1,2,\ldots,n),\quad \omega_i(\nu)>0,\quad \forall\nu\in V$$

If $k\geqslant 0$ is an integer and we have the relation $k\omega_1(\nu_0)=\omega_{i_0}(\nu_0)$, then $\gamma_{i_0}(\nu_0)\neq 0$.

Let us denote by $P^{(i)}$ the projector on the characteristic space of the operator $(-L_\nu)$, corresponding to a pair of eigenvalues $(\lambda_i,\bar\lambda_i)$

$$P^{(i)}u=(u,\psi_i)_H\,\varphi_i+(u,\bar\psi_i)_H\,\bar\varphi_i\equiv y_i,\quad (\varphi_i,\psi_i)_{\bar H}=1$$

where φ_i is an eigenfunction of the operator $(-L_\nu)$, ψ_i is an eigenfunction of the conjugate operator $(-L_\nu^*)$, $(\cdot,\cdot)_H$ is a scalar product in complexification of the real space H. The system of Eqs.(1.4) will now take the form

$$dy_i/dt=-L^{(i)}y_i+N^{(i)}(y_1+y_2+\ldots+y_n+z),\quad i=1,2,\ldots,n \tag{2.1}$$
$$dz/dt=-L_2z+N_2(y_1+y_2+\ldots+y_n+z);\quad L^{(i)}=P^{(i)}L,\ N^{(i)}= P^{(i)}N$$

Let us introduce into the subspaces $P^{(i)}H$ systems of polar coordinate and seek the solution of the system of Eqs.(2.1) in the form

$$y_i=2\mathrm{Re}\,(r_i'\exp(i\theta_i)\,\varphi_i)=2\mathrm{Re}\,(r_i\exp(i\theta_i)\,\varphi_i)+Y_i(r_1,\ldots,r_n,\theta_1,\ldots,\theta_n),\quad z=Z(r_1,\ldots,r_n,\theta_1,\ldots,\theta_n) \tag{2.2}$$

where Z,Y_i are 2π-periodic functions of the variables $\theta_1,\ldots,\theta_n$. We write

$$dr_i/dt=(\gamma_i+b^{(i)})\,r_i,\quad d\theta_i/dt=\omega_i+c^{(i)},\quad i=1,2,\ldots,n \tag{2.3}$$

where $b^{(i)}$, $c^{(i)}$ are functions depending only on the coordinates r_i, θ_i. The dynamic system (2.3) will determine the behaviour of the trajectories of the initial Eq.(2.2) on the $2n$-dimensional IM. Let us assume that

$$\omega_1+c^{(1)}\neq 0,\quad \forall\nu\in V \tag{2.4}$$

Then the phase trajectories of system (2.3) will be determined by a normal system of the form

$$\frac{dr_i}{d\theta_1}=\frac{(\gamma_i+b^{(i)})\,r_i}{\omega_1+c^{(1)}},\quad i=1,2,\ldots,n \tag{2.5}$$
$$\frac{d\theta_i}{d\theta_1}=\frac{\omega_i+c^{(i)}}{\omega_1+c^{(1)}},\quad i=2,3,\ldots,n$$

Let τ denote a fixed instant of time, and let t be the current time. We consider the functions determining the relationship between two points of the trajectories $(r(\tau),\theta(\tau))$ and $(r(t),\theta(t))$

$$r_i(\tau)=R_i^\circ(\theta_1(\tau);r_1(t),\ldots,r_n(t),\ \theta_1(t),\ldots,\theta_n(t)),\quad i=1,2,\ldots,n \tag{2.6}$$
$$\theta_i(\tau)=\Theta_i^\circ(\theta_1(\tau);\ r_1(t),\ldots,r_n(t),\ \theta_1(t),\ldots,\theta_n(t)),\quad i=2,3,\ldots,n$$

The functions R_i°,Θ_i° satisfy system (2.5) and represent autonomous integrals of system (2.3). We shall assume that the functions determining the IM depend on $r_i\ (i=1,2,\ldots,n)$ and $\theta_i\ (i=2,3,\ldots,n)$ through the integrals of motion (2.6)

$$Y_i=Y_i^\circ(\theta_1(t);r_1(\tau),\ldots,r_n(\tau),\theta_2(\tau),\ldots,\theta_n(\tau)) \tag{2.7}$$
$$Z=Z^\circ(\theta_1(t);r_1(\tau),\ldots,r_n(\tau),\ \theta_2(\tau),\ldots,\theta_n(\tau))$$

Substituting (2.2) and (2.7) into Eqs.(2.1) and remembering that Y_i°, Z° depends on t only through the coordinates θ_1, and the derivatives of $r_i(t)$ and $\theta_i(t)$ have the form (2.3), we obtain

$$\omega_1 \frac{\partial Y_i^\circ}{\partial \theta_1} + L^{(i)} Y_i^\circ = -c_t^{(1)} \frac{\partial Y_i^\circ}{\partial \theta_1} + N^{(i)} - 2\,\mathrm{Re}\,[(b_t^{(i)} + i c_t^{(i)}) r_i(t) \exp(i\theta_i(t)) \varphi_i] \tag{2.8}$$

$$\omega_1 \frac{\partial Z^\circ}{\partial \theta_1} + L_2 Z^\circ = -c_t^{(1)} \frac{\partial Z^\circ}{\partial \theta_1} + N_2$$

(the subscript t means that $b_t^{(i)}, c_t^{(i)}$ are functions of $r_i(t)$ and $\theta_i(t)$). Let us put $t = \tau$ in (2.8) and introduce the notation

$$Y_i^\circ(\theta_1(\tau); r_1(\tau), \ldots, r_n(\tau), \theta_2(\tau), \ldots, \theta_n(\tau)) = Y_i^*(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n)$$

$$Z^\circ(\theta_1(\tau); r_1(\tau), \ldots, r_n(\tau), \theta_2(\tau), \ldots, \theta_n(\tau)) = Z^*(r_1, \ldots, r_n, \theta_1, \ldots \theta_n)$$

Now all functions in Eqs.(2.8) will depend on the coordinates r_i, θ_i taken at exactly the same instant τ. We shall write system (2.8) in the form of a single equation for the function

$$g^* = \sum_{i=1}^{n} Y_i^* + Z^*$$

and we have

$$\omega_1 \frac{\partial g^*}{\partial \theta_1} + L_\nu g^* = -c^{(1)} \frac{\partial g^*}{\partial \theta_1} + N - \sum_{i=1}^{n} 2\,\mathrm{Re}\,[(b^{(i)} + i c^{(i)}) r_i \exp(i\theta_i)\, \varphi_i] \tag{2.9}$$

We shall seek the solution of (2.9) in the class of 2π-periodic functions of θ_i, satisfying conditions of orthogonality of the form

$$\int_0^{2\pi} \exp(-i\theta_i)(g^*, \psi_i)_H \, d\theta_i = 0, \quad i = 1, 2, \ldots, n \tag{2.10}$$

It is clear that the set of functions satisfying conditions (2.10) is invariant with respect to the operator $\omega_1 \partial/\partial\theta_1 + L_\nu$. Therefore the right-hand side of Eq.(2.9) must also satisfy conditions (2.10). This demand yields a system of equations for $b^{(i)}, c^{(i)}$

$$b^{(i)} = \frac{1}{2\pi r_i} \mathrm{Re} \int_0^{2\pi} \left(N - c^{(1)} \frac{\partial g^*}{\partial \theta_1}, \psi_i\right)_H \exp(-i\theta_i)\, d\theta_i \tag{2.11}$$

$$c^{(i)} = \frac{1}{2\pi r_i} \mathrm{Im} \int_0^{2\pi} \left(N - c^{(1)} \frac{\partial g^*}{\partial \theta_1}, \psi_i\right)_H \exp(-i\theta_i)\, d\theta_i, \quad i = 1, 2, \ldots, n$$

The following lemma holds (its proof is given in the paper quoted in the previous footnote).

Lemma 1. A number $\rho_1 > 0$ and a set $V_0 \subset V$, can be found such, that when $|r| < \rho_1$ $(|r| = r_1 + \ldots + r_n), \nu \in V_0$, problem (2.9)-(2.11) has a unique solution $g^* \in D$ analytic in r_i, ν and 2π-periodic, of class C^2 in θ_i, and $\|g^*\|_1 = O(|r|^2), r \to 0$. Eqs.(2.11) determine $b^{(i)}(r; \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n), c^{(i)}(r; \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$ uniquely; these are analytic functions of r_i and 2π-periodic function of θ_i of class C^2.

From Lemma 1 it follows that the system of Eqs.(2.9), (2.11) and condition (2.10) determine the projection of Eq.(1.2) on the $2n$-dimensional IM and some auxiliary manifold M_{2n}^* defined by the function g^*. We note that the functions Y_i^* define, in the subspace $P_\nu H$, the change of coordinates $\theta_i' = \theta_i, r_i' = \varphi_i(r, \theta)$.

Let us inspect the form of $2n$-dimensional IM on which the behaviour of the trajectories is determined by the dynamic system (2.3), (2.11). We write

$$g = \sum_{i=1}^{n} Y_i + Z$$

Then, from formulas (2.1)-(2.3) it follows that the IM are determined by the equation

$$dg/dt = -L_\nu g + N - \sum_{i=1}^{n} 2\,\mathrm{Re}\,[(b^{(i)} + ic^{(i)})\, r_i \exp(i\theta_i)\, \varphi_i] \tag{2.12}$$

Let us fix the instant of time $t = \tau$ and seek the solution of Eq.(2.12) satisfying the condition

$$g_\tau(r(\tau), \theta(\tau)) = g^*(r(\tau), \theta(\tau)) \tag{2.13}$$

Let us consider the difference

$$w_\tau(t) = g_\tau(r(t), \theta(t)) - g^*(\theta_1(t); r(\tau), \theta^-(\tau)) \quad (\theta^- = (\theta_2, \ldots, \theta_n))$$

It is clear that $w_\tau(\tau) = 0$. Eqs.(2.12) and (2.9) yield an integral equation for the function $w_\tau(r(t), \theta(t))$, $t \geqslant \tau$

$$\begin{aligned} w_\tau(r, \theta) = \int_{-T}^{0} V(-p)\, \{N\,[Y\,(R\,(p; r, \theta), \Theta\,(p; r, \theta)) + g^*(\Theta_1(p; r, \theta); \\ R\,(-T; r, \theta), \Theta^-(-T; r, \theta)) + \omega_\tau(R\,(p; r, \theta), \Theta\,(p; r, \theta))] - N\,[Y\,(\Theta_1(p; r, \theta); \\ R\,(-T; r, \theta), \Theta^-(-T; r, \theta)) + g^*(\Theta_1(p; r, \theta); R\,(-T; r, \theta), \Theta^-(-T; \\ r, \theta))] - Q\,(R\,(p; r, \theta), \Theta\,(p; r, \theta)) + Q\,(\Theta_1(p; r, \theta); R\,(-T; r, \theta), \\ \Theta^-(-T; r, \theta))\}\, dp \end{aligned} \tag{2.14}$$

$$T = t - \tau, \quad V(t) = \exp(-L_\nu t), \quad Y(r, \theta) = \sum_{i=1}^{n} 2\,\mathrm{Re}\,(r_i \exp(i\theta_i)\, \varphi_i)$$

$$Q(r, \theta) = \sum_{i=1}^{n} 2\,\mathrm{Re}\,[(b^{(i)} + ic^{(i)})\, r_i \exp(i\theta_i)\, \varphi_i]$$

Here $R(s; r, \theta), \Theta(s; r, \theta)$ denote the solutions of system (2.3) with initial conditions $R(0; r, \theta) = r$, $\Theta(0; r, \theta) = \theta$.

Using the principle of compressed mapping and Lemma 1, we can show that Eq.(2.14) has a unique solution, provided that conditions $|r| < \rho_2$, $t \in [\tau, \tau + T_0]$, where $\rho_2 < \rho_1$, T_0 are constant, hold. Differentiating Eq.(2.14) with respect to t and taking into account the fact that $R(-T; r, \theta) = r(\tau)$, $\Theta(-T; r, \theta) = \theta(\tau)$, we obtain $\partial\omega_\tau/\partial t = 0$, i.e. ω_τ depends on time only through the functions $r(t), \theta(t)$. Therefore the solution $\omega_\tau(r(t), \theta(t))$ exists in the whole time interval $[\tau, \tau + T^*]$ on which $|z(t)| < \rho_2$. The function $\omega_\tau(r, \theta)$ belongs to the class C^1 and is a 2π-periodic function of θ_i. Moreover, $\omega_\tau(0, 0) = 0$ and $\|D_{r,\theta}, \omega_\tau\|_1 \leqslant \varepsilon(|r|)$, $\varepsilon(|r|) \to 0$ as $|r| \to 0$ ($D_{r,\theta}\omega_\tau$ is a derivative of the function $\omega_\tau(r, \theta)$).

From what we said above it follows that Eq.(2.12) has a unique solution satisfying condition (2.13)

$$g_\tau(r(t), \theta(t)) = g^*(\theta_1(t);\; r(\tau), \theta^-(\tau)) + \omega_\tau(r(t), \theta(t))$$

Let us use the autonomous integrals (2.6) and denote the parameter $\theta_1(\tau)$ by χ. Then we can assert that the evolutionary Eq.(1.2) has a one-parameter family of LIM $M_{2n}(\chi)$ determined by the function

$$\begin{gathered} g_0(r, \theta; \chi) = g^*(\theta_1; R^\circ(\chi; r, \theta),\; \Theta_2^\circ(\chi; r, \theta), \ldots, \Theta_n^\circ(\chi; r, \theta)) + \\ \omega(r, \theta; \chi) \\ g_0\,|_{\theta_1=\chi} = g^* \end{gathered}$$

It is clear that

$$\|D_{r,\theta} g_0\|_1 \leqslant \varepsilon(|r|); \quad \varepsilon(|r|) \to 0, \quad |r| \to 0$$

Therefore, for sufficiently small $|r|$ the function $Y_0^\circ(r, \theta; \chi) = P_\nu g_0$ will define the mutually single-valued transformation of the coordinates $r \to \varphi(r, \theta)$ is the subspace $P_\nu H$. If, in addition, we take into account that the function g_0 was obtained under the following limitations: $|r| < \rho_2$, $|R^\circ| < \rho_2$, then the domain of existence of LIM will be determined by the inequalities (K is a certain constant)

$$|r| < \rho_0 \leqslant \rho_2, \quad 0 \leqslant \theta_1 - \chi < K, \quad -\infty < \chi < \infty$$

Thus we can formulate the following theorem (it should be noted that an analogous theorem cited in the paper in the previous footnote is not formulated with sufficient accuracy).

Theorem 2. Let conditions 1°-6° hold. Then Eq.(1.2) will have, for the value $\nu \in V_0 \subset V$, a one-parameter family of two-dimensional LIM'S

$$M_{2n}(\chi) = \{(y_1, \ldots, y_n, z);\; y_i = 2\mathrm{Re}\,(r_i \exp(i\theta_i)\, \varphi_i) + Y_{0i}^\circ(r, \theta; \chi)\},$$

$$z = Z_0^\circ(r, \theta; \chi)\} \quad (-\infty < \chi < \infty)$$

The functions $Y_{0i}^\circ(r, \theta; \chi)$, $Z_0^\circ(r, \theta; \chi)$ belong to the class C^1 and are defined for $|r| < \rho_0$, $0 \leqslant \theta_1 - \chi < K$ (ρ_0, K are constants).

In addition we have

$$Y_{0i}^\circ(0, \theta; \chi) = D_{r,\theta} Y_{0i}^\circ(0, \theta; \chi) = Z_0^\circ(0, \theta; \chi) = D_{r,\theta} Z_0^\circ(0, \theta; \chi) = 0$$

The functions Y_{0i}°, Z_0° satisfy the conditions

$$Y_{0i}^\circ|_{\theta_1=\chi} = Y_i^*(r, \theta), \quad Z_0^\circ|_{\theta_1=\chi} = Z^*(r, \theta)$$

where Y_i^*, Z^* are solutions of problem (2.19)-(2.11) depending analytically on the coordinates r_i and the parameter ν, and 2π-periodic functions of θ_i of class C^2.

The trajectories of Eq.(1.2) are defined on the manifolds $M_{2n}(\chi)$ by a unique system of Eqs.(2.3) where $b^{(i)}(r; \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$, $c^{(i)}(r; \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$ are analytic functions of r_i, while ν and 2π are periodic functions of θ_i of class C^2.

Any global solution $u(t)$ of Eq.(1.2) in which

$$\|P_\nu u(t)\|_1 < \min(\rho_0, \rho), \quad t \in [0, \infty); \quad \|(I - P_\nu) u(0)\|_1 < \rho$$

$$\rho + \rho^{1/\alpha} < \begin{cases} b_1(\varkappa_\nu - q_\nu) & (q_\nu \geqslant 0) \\ b_2 \varkappa_\nu & (q_\nu < 0) \end{cases}$$

will be attracted by the manifolds $M_{2n}(\chi)$.

Corollary. The manifold M_{2n}^* is locally attractive.

Proof. Let us consider the trajectory $u(t)$ of Eq.(2.2) satisfying the conditions

$$\|P_\nu u(t)\|_1 < \min(\rho_0, \rho), \; t \in [0, \infty); \; \|(I - P_\nu) u(0)\|_1 < \rho$$

Let $\{t_k\}$ denote an increasing sequence of values of t: $0 < t_1 < \ldots < t_k < \ldots$ We have the corresponding sequence of points of the trajectory $\{u_k\}$: $u_k = u(t_k)$. Let us denote by $(r^{(k)}, \theta^{(k)})$ the coordinates of the projection u_k on the limit manifold M_{2n}^*. From relation (2.2) it follows that

$$\theta_i^{(k)} = \operatorname{arctg}[\operatorname{Im}(u_k, \psi_i)_H / \operatorname{Re}(u_k, \psi_i)_H], \; i = 1, 2, \ldots, n$$

and the coordinates $r_i^{(k)}$ are determined by a system of equations of the form

$$r_i^{(k)} + |(Y_i^*(r^{(k)}, \theta^{(k)}), \psi_i)_H| = |(u_k, \psi_i)_H|, \; i = 1, 2, \ldots, n$$

Let us consider the LIM $M_{2n}(\theta_1^{(k)})$ for some value of k. Since $M_{2n}(\theta_1^{(k)})$ is a locally attractive manifold, it follows that for sufficiently large values of t we have

$$\|z(t) - Z_0^\circ(r(t), \theta(t); \theta_1^{(k)})\|_1 \leqslant \text{const}\cdot\exp(-\gamma t) \tag{2.15}$$

$$\theta_1(t) \geqslant \theta_1^{(k)}; \; z(t) = (I - P_\nu) u(t)$$

where $r(t)$, $\theta(t)$ are the coordinates of the projection of $u(t)$ onto the manifold $M_{2n}(\theta_1^{(k)})$.

From (2.15) it follows that the following inequality holds for fairly large values of k:

$$\|z(t_k) - Z_0^\circ(r(t_k), \theta(t_k); \theta_1^{(k)})\|_1 \leqslant \text{const}\cdot\exp(-\gamma t_k)$$

Since $\theta_i(t_k) = \theta_i^{(k)}$, then $r_i(t_k) = r_i^{(k)}$ and $Z_0^\circ(r(t_k), \theta(t_k); \theta_1^{(k)}) = Z^*(r^{(k)}, \theta^{(k)})$. Therefore

$$\|z(t_k) - Z^*(r^{(k)}, \theta^{(k)})\|_1 \leqslant \text{const}\cdot\exp(-\gamma t_k), \; k \to \infty$$

Since ω are the limit points, u_p are determined by the relation $\lim u(t) = u_p$ as $t \to \infty$ and any point on the closed trajectory is an ω-limit point, it follows that all ω-limit points and periodic solutions of Eq.(1.2) from the sufficiently close neighbourhood of the zero will belong to the limit manifold M_{2n}^*.

In order to determine the $2n$-dimensional invariant projection of Eq.(2.2) and the limit manifold M_{2n}^*, we shall write the functions $g^*(r, \theta)$, $b^{(i)}(r, \theta)$, $c^{(i)}(r, \theta)$ in the form of series in powers of r_i

$$g^* = \sum_{|S|=2}^{\infty} g_S r^S, \quad r^S = r_1^{s_1} \cdot r_2^{s_2} \ldots r_n^{s_n}, \quad |S| = s_1 + \ldots + s_n \tag{2.16}$$

$$b^{(i)} = \sum_{|S|=1}^{\infty} b_S^{(i)} r^S, \quad c^{(i)} = \sum_{|S|=1}^{\infty} c_S^{(i)} r^S$$

We substitute (2.16) into (2.9), (2.10) and (2.11) and equate the cofficients of like powers of r^S. This yields the following recurrence system of linear equations for g_S:

$$\omega_1 \frac{\partial g_S}{\partial \theta_1} + L_\nu g_S = -\sum_{i=1}^{n} 2\,\mathrm{Re}\,[(b_{S_i}^{(i)} + ic_{S_i}^{(i)}) \exp(i\theta_i)\,\varphi_i] - \sum_{K+P=S} c_K^{(1)} \frac{\partial g_P}{\partial \theta_1} + N_S, \tag{2.17}$$

$$N = \sum_{|S|=2}^{\infty} N_S r^S, \quad S_i = (s_1, \ldots, s_i - 1, \ldots, s_n)$$

The functions g_s, periodic in θ_i, must satisfy the conditions

$$\int_0^{2\pi} \exp(-i\theta_i)\,(g_S, \psi_i)_H\, d\theta_i = 0 \tag{2.18}$$

and the functions $b_{S_i}^{(i)}(\theta)$, $c_{S_i}^{(i)}(\theta)$ are uniquely defined by the equation

$$b_{S_i}^{(i)} + ic_{S_i}^{(i)} = \frac{1}{2\pi} \int_0^{2\pi} \exp(-i\theta_i) \Big(N_S - \sum_{K+P=S} c_K^{(1)} \frac{\partial g_P}{\partial \theta_1}, \psi_i\Big)_H d\theta_i \tag{2.19}$$

3. Two-dimensional invariant of the Navier-Stokes equations for Poiseuille flow. Let us consider the flow of a liquid in a plane channel. Let the x axis be directed along the channel and let the dimensionless laminar velocity profile have the form $U = 1 - y^2$. When the perturbations are two-dimensional, it is convenient to convert from the Navier-Stokes equations to an equation for the stream function of the perturbation

$$\frac{\partial \Delta\Psi}{\partial t} + U \frac{\partial \Delta\Psi}{\partial x} - U'' \frac{\partial \Psi}{\partial x} - \frac{1}{R} \Delta^2\Psi - \frac{\partial \Psi}{\partial x} \frac{\partial \Delta\Psi}{\partial y} + \frac{\partial \Psi}{\partial y} \frac{\partial \Delta\Psi}{\partial x} = 0 \tag{3.1}$$

We shall consider solutions of Eq.(3.1) periodic in x. Then the part of Ω will be played by the periodicity cell

$$\Omega = \{x, y\colon\ 0 \leqslant x \leqslant 2\pi/\alpha,\ -1 \leqslant y \leqslant 1\}$$

and the stream function Ψ will have to satisfy the following boundary conditions:

$$\Psi\,|_{y=\pm1} = \partial\Psi/\partial y\,|_{y=\pm1} = 0, \quad \Psi\,(x + 2\pi/\alpha, y, t) = \Psi\,(x, y, t) \tag{3.2}$$

The eigenvalues of the corresponding linear operator $(-L_\nu)$ are found from an equation of the form

$$\lambda\Delta\varphi + U\partial\Delta\varphi/\partial x - U''\partial\varphi/\partial x - R^{-1}\Delta^2\varphi = 0 \tag{3.3}$$

and boundary conditions (3.2).

In order to construct a two-dimensional invariant projection of problem (3.1), (3.2) we take as σ_1 a pair of first eigenvalues of problem (3.3), (3.2) (the ordering is carried out according to the magnitude of $\mathrm{Re}\,\lambda$). Using the representation

$$\varphi\,(x, y) = \exp\,(-i\alpha x)\, f\,(y) \tag{3.4}$$

we obtain for the function $f(y)$ an equation which is the complex conjugate of the Orr-Sommerfeld equation

$$-i\alpha\,[(U - \bar{c})\,(d^2/dy^2 - \alpha^2)\, f - U''f] - R^{-1}\,(d^2/dy^2 - \alpha^2)^2 f = 0 \tag{3.5}$$

$$\bar{c} = -\gamma/\alpha + \omega/\alpha$$

(the minus sign in the exponential function in (3.4) is chosen in order to ensure that the quantity ω is positive in accordance with condition 9°). We know that an even eigenfunction $f(y)$ corresponds to the first eigenvalue of the Orr-Sommerfeld equation; therefore it must satisfy the following boundary conditions:

$$f\,(-1) = f'\,(-1) = f'\,(0) = f'''\,(0) = 0$$

In the present case the first manifold M_2^* is determined by the function $g^*(r, \theta)$, depending only on two variables, r and θ. The functions b and c determining the invariant projection depend only on the coordinate r. It can be shown that in (2.16)

$$g_s = \sum_{k=-s}^{s} g_{sk} \exp(ik\theta^*), \quad \theta^* = \theta - \alpha x$$

and

$$b(r) = \sum_{n=1}^{\infty} b_{2n} r^{2n}, \quad c(r) = \sum_{n=1}^{\infty} c_{2n} r^{2n}$$

From (2.17) we obtain the following system of recurrence equations for determining g_{sk} (δ_{ki} is the Kronecker delta):

$$ik[(\omega - \alpha U)\Delta_k + \alpha U'']g_{sk} - R^{-1}\Delta_k^2 g_{sk} = -\delta_{k1}(b_{s-1} + ic_{s-1})\Delta_1 f - \delta_{k,-1}(b_{s-1} - ic_{s-1})\Delta_1 f - ik \sum_{q+p=s} c_q \Delta_k g_{pk} - i\alpha \sum_{q+p=s} \sum_{l+j=k} \times \left[l g_{ql} \frac{d}{dy} \Delta_j g_{pj} - j \frac{dg_{ql}}{dy} \Delta_j g_{pj} \right] \tag{3.6}$$

$$\Delta_k g_{sk} = (d^2/dy^2 - (k\alpha)^2) g_{sk}$$

The functions g_{sk} must satisfy the following boundary conditions:

$$g_{sk}(-1) = g_{sk}'(-1) = 0$$

$g_{sk}'(0) = g_{sk}'''(0) = 0$, if s is even, and $g_{sk}(0) = g_{sk}''(0) = 0$, if s is odd.
We have the following condition of the form:

$$\int_{-1}^{1} f^* \Delta_1 g_{s1}\, dy = 0$$

corresponding to conditions of orthogonality (2.18), where f^* is the eigenfunction of the equation conjugated to (3.5).

The coefficients b_{s-1}, c_{s-1} are determined from the equation

$$\int_{-1}^{1} F_{s1} f^*\, dy = 0$$

where F_{s1} is the right-hand side of Eq.(3.6) for the function g_{s1}.

Solving Eqs.(3.6) in succession we obtain the limit manifold M_2^*, and two-dimensional dynamic system describing the behaviour of the trajectories on the family of IM'S

$$dr / dt = (\gamma + b(r))\, r, \quad d\theta / dt = \omega + c(r) \tag{3.7}$$

The behaviour of the perturbations as $t \to \infty$, which is determined by two critical trajectories of system (3.7), represents the greatest interest. Examples of such trajectories are the limit cycles, and periodic solutions of Eq.(3.1) correspond to them. The amplitudes of the limit cycles are found from the condition

$$\gamma + b(r) = 0 \tag{3.8}$$

Taking into account the fact that $b(r)$ is an even function, we introduce the notation

$$p = r^2, \quad B_0 = \gamma(R, \alpha), \quad B_n = b_{2n}(R, \alpha)$$

$$F(p; R, \alpha) = \sum_{n=0}^{\infty} B_n p^n, \quad F_N(p; R, \alpha) = \sum_{n=0}^{N} B_n p^n$$

We solved numerically the approximate equation for the amplitudes

$$F_N(p; R, \alpha) = 0 \tag{3.9}$$

for $N \leqslant 5$. The calculation showed that, depending on the behaviour of solutions of (3.9), we can single out four domains of values of the wave number: α: $I_1 = (\alpha\colon \alpha \geqslant 0.98)$, $I_2 = (\alpha\colon 0.92 < \alpha < 0.98)$, $I_3 = (\alpha\colon 0.9 < \alpha \leqslant 0.92)$, $I_4 = (\alpha\colon \alpha \leqslant 0.9)$.

A typical plot of the solutions of (3.9) for $2 \leqslant N \leqslant 5$ and $\alpha \in I_1$ ($\alpha = 1$) is shown in Fig.1 (R_0 is the linear neutral Reynold number). When $p < p_*(\alpha)$, all approximations yield similar solutions which correspond to the amplitudes of the unstable, selfexcited oscillatory modes of flow. When $p > p_*(\alpha)$, the pattern changes qualitatively. If, in the odd approximations, the solution is continued without break into the region of smaller R, then in the even approximations a second solution appears, corresponding to a stable selfexcited oscillation. The solution merges with the first solution at some critical value of $R = R_*(\alpha)$.

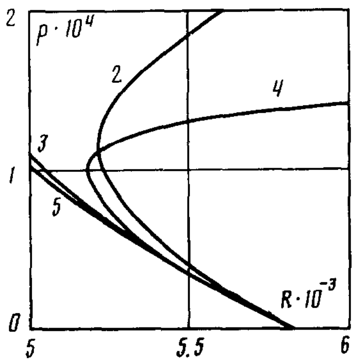

Fig.1

We note here that the stability of selfexcited oscillation with the amplitude is determined by the sign of the derivative $\partial F/\partial p\,|_{p_0}$ $(p_0 = r_0^2)$. The selfexcited oscillation will be stable if $\partial F/\partial p\,|_{p_0} < 0$, and unstable when $\partial F/\partial p\,|_{p_0} > 0$.

When the values of α lie in the interval I_3, the graphs of the solutions will be analogous to those in Fig. 1, but the even and odd approximations will be interchanged. The odd approximations will have two solutions, and the even approximations a single solution. The region I_2 will become transitional, and we can have here two solutions of approximations of different parity. In the region I_4 the pattern will become more complicated.

We shall use the methods of catastrophe theory /19/ to analyse the behaviour of the solution of Eq.(3.8) when $p \geqslant p_*$.

We shall consider F as a two-parameter family of functions $F(p;\, R,\, \alpha)$, and inspect the set of zeros of this family. We shall begin from the region I_1. Let $p^*(\alpha)$, $R^*(\alpha)$ be the values of p and R such, that when $p < p^*(\alpha)$, $R > R^*(\alpha)$, Eq.(3.8) will have a unique solution to which the solutions of Eqs.(3.9) will converge, while when $p > p^*$, the approximations $F_N(p;\, R^*,\, \alpha)$ will diverge. It is clear that the quantities p_*, R_* will yield approximate values of p^* and R^* respectively. Let us fix some value of $\alpha_0 \in I_1$. We can assume that $B_n(R^*(\alpha_0),\, \alpha_0) \neq 0$ $(n = 1, 2, \ldots)$, since these conditions hold for almost every value of α.

Let us consider the family $F(p;\, R,\, \alpha)$ in the following region: $0 \leqslant p < p^*(\alpha_0)$, $(R, \alpha) \in \Delta(p)$ where Δ is the neighbourhood of $(R^*(\alpha_0), \alpha_0)$, in which the coefficients B_n have constant sign and the approximations $F_N(p;\, R,\, \alpha)$ converge. Numerical computations have shown that for any $\alpha \in I_1$ and R from some neighbourhood $R_*(\alpha)$ containing $R_0(\alpha)$, the coefficients B_1, B_3, B_5 are positive and B_2, B_4 are negative. Therefore $F(p;\, R,\, \alpha)$ can be written in the form

$$F = B_0 + y_1 - y^2 \quad (B_0 \leqslant 0,\ B_0(R_0(\alpha), \alpha) = 0)$$
$$y_1(p) = \sum_{k=1}^{K_1} B_{i_k}^{+} p^{i_k}, \quad y(p) = \Big(\sum_{k=1}^{K_2} |B_{j_k}^{-}|\, p^{j_k}\Big)^{1/2}$$

$(K_1, K_2$ may become $\infty)$; $B_{i_k}^{+}$ are positive coefficients belonging to $\{B_n\}_1^{\infty}$, and $B_{j_k}^{-}$ are negative. The analytic functions $y_1(p;\, R,\, \alpha)$ and $y(p;\, R,\, \alpha)$ satisfy the conditions

$$y_1(0;\ R,\ \alpha) = y(0;\ R,\ \alpha) = 0;\ \partial y_1/\partial p,\ \partial y/\partial p > 0$$

It can be shown that there exist C^{∞}-continuations y_1°, y° of the functions y_1 and y, with the following conditions holding in some neighbourhood of the point $(p^*(\alpha_0),\, R^*(\alpha_0),\, \alpha_0)$:

$$\partial y_1^{\circ}/\partial p > 0,\ \partial y^{\circ}/\partial p > 0;\ 0 \leqslant p < p_1,\ p_1 > p^*\ (R, \alpha) \in \Delta \tag{3.10}$$

Let us consider the family $F^{\circ} = B_0 + y_1^{\circ} - (y^{\circ})^2$. By virtue of condition (3.10) there exists a smooth change of coordinates $p \to y^{\circ}(p)$ and $F^{\circ}(p;\, R,\, \alpha)$ equivalent to the family

$$\Phi(y^{\circ};\ R,\ \alpha) = B_0(R,\ \alpha) + y_1^{\circ}(y^{\circ};\ R,\ \alpha) - (y^{\circ})^2$$

Let $y_0(R,\, \alpha)$ be a solution of the equation $\Phi(y^{\circ}) = 0$ and let us consider the quantity

$$\frac{\partial \Phi}{\partial y^{\circ}}\Big|_{y_0} = \varkappa(y_0) - 2y_0, \quad \varkappa(y_0) = \frac{\partial y_1^{\circ}}{\partial p}\,\frac{\partial p}{\partial y^{\circ}}(y_0)$$

Since $\varkappa(y_0) > 0$ for all y_0, it follows that for sufficiently small y_0 the family Φ has simple zeros, with the corresponding simple zeros of the family $F(p;\, R,\, \alpha)$. However, when y_0 increases, the derivative $\partial\Phi/\partial y^{\circ}|_{y_0}$ may vanish and a multiple zero will appear in the family $F(p;\, R,\, \alpha)$.

It is natural to assume that the divergence of the solutions of Eqs.(3.9) is related to this aspect, i.e.

$$\frac{\partial y_1^{\circ}}{\partial p}\,\frac{\partial p}{\partial y^{\circ}}(y_0;\, R,\, \alpha) - 2y_0 = 0 \tag{3.11}$$
$$R = R^*(\alpha_0),\ y_0 = y^* = y^0(p^*,\ R^*,\ \alpha_0)$$

We can assume here that

$$\frac{\partial^2\Phi}{(\partial y^\circ)^2}(y^*; R^*, \alpha_0) = -\left(2 - \frac{\partial^2 y_1^\circ}{(\partial y^\circ)^2}(y^*, R^*, \alpha_0)\right) \neq 0 \tag{3.12}$$

so that the condition will hold for almost all values of α.

We see that in some neighbourhood of the point (p^*, R^*, α) the family Φ will be equivalent to the family

$$\Phi_1(u;\ t_1,\ t_2) = t_1 + t_2 u + \beta(t_1, t_2)\, u^2 + f(u;\ t_1,\ t_2)$$
$$u = y^\circ - y^*,\ t_1(R, \alpha) = \Phi(y^*; R, \alpha),\ t_2(R, \alpha) = \frac{\partial\Phi}{\partial y^\circ}(y^*; R, \alpha)$$
$$\beta(t_1, t_2) = \frac{1}{2}\frac{\partial^2\Phi}{(\partial y^\circ)^2}(y^*; R(t_1, t_2), \alpha(t_1, t_2))$$
$$f(0; t_1, t_2) = \frac{\partial f}{\partial u}(0; t_1, t_2) = \frac{\partial^2 f}{\partial u^2}(0; t_1, t_2) = 0,\quad t_1(R^*, \alpha_0) = t_2(R^*, \alpha_0) = 0$$

Using Theorem 8.6 of /19/, we can show that $\Phi_1(u; t_1, t_2)$ represents the versal deformation of the 2-defined function $\beta(0, 0)\, u^2 + f(u;\ 0,\ 0)$, having a codimensionalty of 1 (the codimensionality is determined in the space of all polynomials of u). Therefore, the catastrophe of the fold will represent the universal deformation. It follows that in the region I_1 the correct description of solutions of Eqs.(3.8) will yield the even approximations (3.9). We note that near the point (p_*, R_*, α_0) the family $F_2(p; R, \alpha)$ can be reduced to a form analogous to Φ_1, with $f \equiv 0$.

Let us now transfer to the region I_2. The signs of the coefficients B_3, B_4, B_5 will change in the interval $0.96 \leqslant \alpha < 0.98$ near the critical points $(p_*(\alpha), R_*(\alpha), \alpha)$. It is clear from the previous analysis that the character of the singularities $F(p; R, \alpha)$ will remain unchanged and only the form of the functions $y_1(p), y(p)$ will be different. Let us consider the interval $0.92 < \alpha < 0.96$. Here the coefficient B_2 changes its sign, B_3, B_5 are negative and B_1, B_4 are positive. Let us write $F(p;\ R,\ \alpha)$ in the form

$$F = B_0 + y_1(p) + B_2 p^2 - y^3(p)$$

where y_1 is the sum of positive terms of the series $F(p)$ and $(-y^3)$ is the sum of negative terms. We can now show that the set of zeros of the family $F(p; R, \alpha)$ has, as before, a fold-type singularity. In the region I_3 we have $B_1, B_2, B_4 > 0$ and $B_3, B_5 < 0$. It can be shown that the critical points $(p^*(\alpha), R^*(\alpha), \alpha)$ will also be folds, and a correct description is given by the solutions of Eqs.(3.9) for $N = 3$ and $N = 5$.

On passing from region I_3 to region I_4 we find that the coefficient B_1 changes its sign near the linear neutral curve $(R = R_0(\alpha);\ B_0 = 0)$. Let us fix a value of $\alpha = \alpha_0$, near to $\alpha = 0.9$. It can be shown that in some neighbourhood of the point $(p^*(\alpha_0), R^*(\alpha_0), \alpha_0)$ the family $F(p, R, \alpha)$ is equivalent to the family $\Phi(y^\circ; R, \alpha)$ of the form

$$\Phi = B_0 + B_1 p(y^\circ) + (y_1^\circ(y^\circ))^2 - (y^\circ)^3$$

where $y_1^\circ(p),\ y^\circ(p)$ is a smooth continuation of the functions

$$y_1(p) = \Big(\sum_{\substack{k=1\\ i_k \neq 1}}^{K_1} B_{i_k}^+ p^{i_k}\Big)^{1/2},\quad y(p) = \Big(\sum_{\substack{k=1\\ j_k \neq 1}}^{K_2} |B_{j_k}^-|\, p^{j_k}\Big)^{1/3}$$

When the values of R are close to $R_0(\alpha_0)\ (\alpha_0 \neq 0.9)$, the equation $\Phi(y^\circ) = 0$ has a small solution $p \approx -B_0/B_1$. Since the quantity $p = r^2$ must be positive, a solution for $\alpha_0 > 0.9$ will exist when $R \geqslant R_0(\alpha_0)\ (B_0 \leqslant 0)$, and for $\alpha_0 < 0.9$ when $R \geqslant R_0(\alpha_0)\ (B_0 \geqslant 0)$, i.e. at the point $\alpha = 0.9$ the subcritical bifurcation will become supercritical.

Let us consider the quantity

$$\frac{\partial\Phi}{\partial y^\circ}(y_0) = \left(B_1 \frac{\partial p}{\partial y^\circ}(y_0) + 2y_1^\circ(y_0)\frac{\partial y_1^\circ}{\partial y^\circ}(y_0) - 3y_0^2\right)$$

where y_0 is a solution of the equation $\Phi(y^\circ) = 0$. We see that when $\alpha_0 < 0.9$ the family Φ, has, apart from the critical point y^*, arriving from the region I_3, another critical point

$$y^{**} \approx -{}^1\!/_2 B_1\, |B_3|^{1/3} / B_2$$

which appears when the character of bifurcation changes. The second fold of the set of zeros of the family $F(p; R, \alpha)$ corresponds to this point.

Fig.2 shows graphically the solution of Eq.(3.9) $(N = 3)$ for $\alpha = 0.91, 0.9, 0.895$ (curves *1-3* respectively). We see that for $\alpha \leqslant 0.9$ there exist two folds which approach each other as α decreases. Numerical computations showed that the folds coalesce at $\alpha = \alpha^* \approx 0.89$, and only a single solution of Eq.(3.9) exists at $\alpha < \alpha^*$. The corresponding analysis using the method of catastrophe theory confirms that the set of zeros $F(p; R, \alpha)$ has an assembly-type singularity at the point (p^*, R^*, α^*).

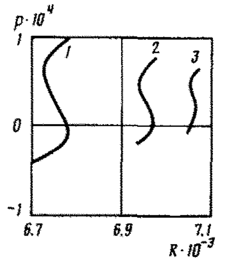

Fig.2

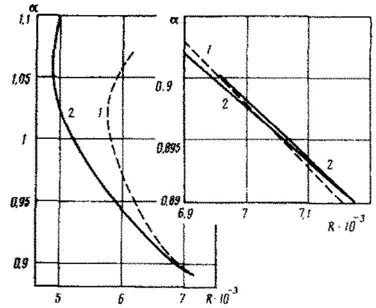

Fig.3

Thus we have shown that two subcritical periodic solutions of Eq.(3.1) exist for $\alpha \in I_1 \cup I_2 \cup I_3$, the unstable and the stable (in the order of increasing amplitudes), and merging at the fold point $(p^*(\alpha), R^*(\alpha), \alpha)$. This confirms the result obtained earlier by the asymptotic method in /6/ and by direct numerical computations in /20/. Moreover, we have established here that at the point at which the subcritical bifurcation becomes supercritical, we have a bifurcation of the generation of two periodic solutions. As a result, another fold appears at the amplitude surface of the selfexcited oscillations and three periodic solutions exist in the region $0.9 > \alpha > 0.89$, the stable, unstable and stable. The solutions merge at the assembly point when $\alpha^* \approx 0.89$, $R^* \approx 7173$.

Fig.3 shows the result of a numerical computation of the fold curve (*1* depicts the neutral linear curve and *2* the folds curve).

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